Velocity Quantization Approach of the One-Dimensional Dissipative Harmonic Oscillator

G. López[1] and P. López[1]

Given a constant of motion for the one-dimensional harmonic oscillator with linear dissipation in the velocity, the problem to get the Hamiltonian for this system is pointed out, and the quantization up to second order in the perturbation approach is used to determine the modification on the eigenvalues when dissipation is taken into consideration. This quantization is realized using the constant of motion instead of the Hamiltonian.

KEY WORDS: constant of motion; Hamiltonian; quantization; harmonic oscillator; dissipation.
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1. INTRODUCTION

To find the Lagrangian and the Hamiltonian from the equations of motion of a given system (so called "inverse problem of the calculus of variations" or "the inverse problem of the mechanics") is not so trivial as one could think at first sight, even for one-dimensional systems and despite their existence is guaranteed here (Darboux, 1984). If the system is autonomous (the forces do not depend explicitly on time) or nonautonomous (total force depend explicitly on time), it is natural to try to find Lagrangians and Hamiltonians which do not depend explicitly on time or which depend explicitly on time for the latter. One possible approach to find these quantities for a one-dimensional system is to find first a constant of motion of the system, and then to obtain the Lagrangian and Hamiltonian (Kobussen, 1979; Leubner, 1987; López, 1996a; Yan, 1981). This constant of motion, of course, is chosen to be independent explicitly on time for autonomous systems, and explicitly depending on time, for nonautonomous systems. It has been shown that when one tries to get quantities (constant of motion, Lagrangian or Hamiltonian) which are explicitly depending on time for autonomous systems,

[1] Departamento de Física de la Universidad de Guadalajara, Apartado Postal 4-137, 44410 Guadalajara, Jalisco, México; e-mail: gulopez@udgserv.cencar.udg.mx, plopez@ccip.udg.mx.

there may be a concern about ambiguities (López, 1998) or about consistence on the mathematical quantities (López and Hernández, 1989).

However, one common procedure that has been done is to guess an explicitly time depending Hamiltonian for the harmonic oscillator with exponential time dependence of its mass (Caldirola, 1941; Kanai, 1948; Yeon *et al.*, 1987) which brings about the one-dimensional harmonic oscillator with linear dissipation in the velocity. Indeed, there has been a lot of studies on the quantization of this Hamiltonian (Colegrave and Abdalla, 1981; Dekker, 1977; Dekker, 1981; Dodonov *et al.*, 1981; Leach, 1983) and decoherence of this quantum system (Rosenau da Costa *et al.*, 2000). However, since the one-dimensional harmonic oscillator with linear dissipation on the velocity is an autonomous system, one should get time independent dynamical quantities (constant of motion, Lagrangian or Hamiltonian) to describe it. After this, one should proceed to make the quantization of this system in a consistent way. In fact, an explicitly time independent constant of motion for this system has been found already (López, 1996b). So, in this paper, we will study the classical damping behavior of the system with this constant of motion, and we point out the difficulties to get its associated Hamiltonian. Finally, we study its quantization using the same constant of motion and the velocity operator. This last study is done up to second order in perturbation theory and for weak dissipation.

2. CONSTANT OF MOTION AND LAGRANGIAN

The one-dimensional harmonic oscillator with linear dissipation on the velocity is described by the equation

$$m\ddot{x} + \alpha x + kx = 0, \tag{1}$$

where m is the mass of the particle, α is the dissipative constant, k is the spring constant, x is the particle position, and $\dot{x}$ and $\ddot{x}$ are its first and second differentiation with respect the time. By defining the new variable $\dot{x} = v$, Eq. (1) can be written as the following autonomous dynamical system

$$\dot{x} = v, \qquad \text{and} \qquad \dot{v} = -\omega^2 x - 2\omega_\alpha v, \tag{2}$$

where $w = \sqrt{k/m}$ is the natural angular frequency of the harmonic oscillator without dissipation, and $\omega_\alpha = \alpha/2m$ is the new dissipation parameter. A constant of motion for the system (2) is a function $K = K_{(x,v)}$ which satisfies the following partial differential equation (John, 1974; López, 1999)

$$v\frac{\partial K}{\partial x} - (\omega^2 x + 2\omega_\alpha v)\frac{\partial K}{\partial v} = 0. \tag{3}$$

The solution, K_α, of this equation which satisfies the following limit $\lim_{\alpha\to 0} K_\alpha = mv^2/2 + m\omega_\alpha x^2/2$ is given by (Caldirola, 1941)

$$K_\alpha(x, v) = \frac{m}{2}(v^2 + 2\omega_\alpha xv + \omega^2 x^2)e^{-2\omega_\alpha G(v/x,\omega,\omega_\alpha)}, \tag{4a}$$

where the function G has been defined as

$$G(v/x, \omega, \omega_\alpha) = \begin{cases} \dfrac{1}{2\sqrt{\omega_\alpha^2 - \omega^2}} \ln\left[\dfrac{\omega_\alpha + v/x - \sqrt{\omega_\alpha^2 - \omega^2}}{\omega_\alpha + v/x + \sqrt{\omega_\alpha^2 - \omega^2}}\right] & \text{if } \omega^2 < \omega_\alpha^2 \\[4mm] \dfrac{1}{\omega_\alpha + v/x} & \text{if } \omega^2 = \omega_\alpha^2 \\[4mm] \dfrac{1}{\sqrt{\omega^2 - \omega_\alpha^2}} \arctan\left(\dfrac{\omega_\alpha + v/x}{\sqrt{\omega^2 - \omega_\alpha^2}}\right) & \text{if } \omega^2 > \omega_\alpha^2 \end{cases} \tag{4b}$$

For weak dissipation, one has the following expression for the constant of motion

$$K = \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2 x^2 + \frac{m\omega_\alpha}{\omega}\left[xv\omega - (v^2 + \omega^2 x^2)\arctan\left(\frac{v}{wx}\right)\right]. \tag{5}$$

The Lagrangian for the system (1) can now be constructed from the known expression (Kobussen, 1979)

$$L(x, v) = v \int^v \frac{K(x, \xi)d\xi}{\xi^2} . \tag{6}$$

For the general case (4), it is not possible to get a close expression for the Lagrangian, but for the weak dissipation case (5), one gets

$$L = \frac{1}{2}mv^2 - \frac{1}{2}m\omega^2 x^2 + \frac{m\omega_\alpha}{\omega}\left[(\omega^2 x^2 - v^2)\arctan\left(\frac{v}{\omega x}\right)\right.$$
$$\left. + \omega xv \ln\left(\frac{\omega^2 x^2 + v^2}{\omega^2 x^2}\right)\right]. \tag{7}$$

The generalized linear momentum ($p = \partial L/\partial v$) is

$$p = mv + \frac{w\omega_\alpha}{\omega}\left[\omega x + x\omega \ln\left(\frac{\omega^2 x^2 + v^2}{\omega^2 x^2}\right) - 2v \arctan\left(\frac{v}{\omega x}\right)\right]. \tag{8}$$

To get the Hamiltonian, it is necessary from (8) to express the variable v as a function of the variables x and p, $v = v(x, p)$. In this way, one makes the substitution of this variable on the Legendre transformation, $H(x, p) = v(x, p)p - L[(x, v(x, p)]$, or in the constant of motion, $H(x, p) = K[x, v(x, p)]$. However, one notices immediately form (8) that it is not possible to do this. Thus, the Hamiltonian can not be given explicitly but implicitly through the constant of motion (5).
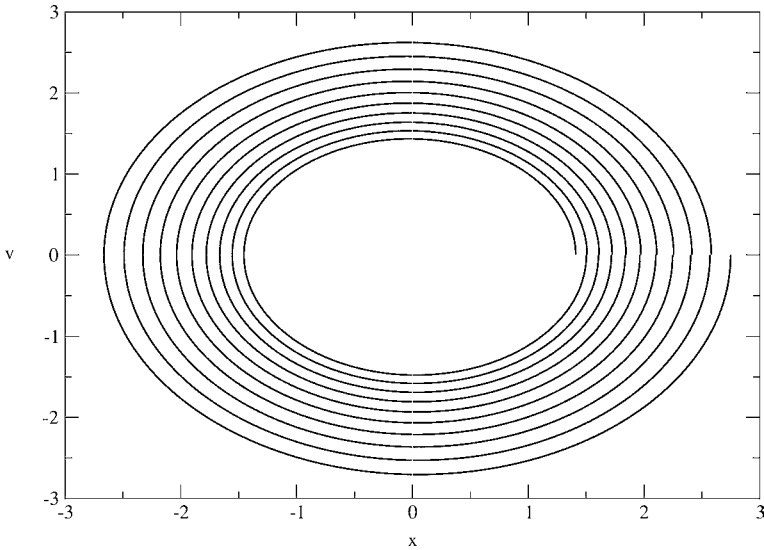
Fig. 1. Trajectory on the phase space (x, v) as determinate by (4), where $m = 1\ K\ gr$, $w = 1\ \sec^{-1}$ and $\omega_\alpha = 0.001\ \sec^{-1}$.

A trajectory in the phase space (x, v) can be seen on Fig. 1, where the constant of motion (5) has been used. Of course, in order to keep the continuity at $(v = 0, x < 0)$, the arctan function changes its value due to the multivalue functions. This means that our constant of motion is really a local constant of motion (which is valid on the half plane $v < 0$ or $v > 0$), and it changes its value every time the trajectory crosses the line $v = 0$. This number of crossing is a numerable set. Therefore, this set has measure zero (Hewitt and Stromberg, 1965). In this way, one can say that (4) or (5) represents a constant of motion almost everywhere in the phase space.

Let us now change the variables (x, v) by a new variables (ϕ, J) defined as

$$\phi = \arctan\left(\frac{v}{\omega x}\right), \quad \text{and} \quad J = \frac{m}{2\omega}(v^2 + \omega^2 x^2), \qquad (9a)$$

where the inverse transformation is given by

$$x = \sqrt{\frac{2J}{m\omega}}\cos\phi, \quad \text{and} \quad v = \sqrt{\frac{2\omega J}{m}}\sin\phi. \qquad (9b)$$

The dynamical system (2) is then transformed to the system

$$\dot{\phi} = -\left(\omega + \frac{2\omega_\alpha \tan\phi}{1 + \tan^2 \phi}\right) \qquad (10a)$$

and

$$\dot{J} = -4\omega_\alpha J \, \sin^2 \phi. \tag{10b}$$

These equations are readily solved, and their solutions are

$$\phi(t) = -\arctan\left[\tan(\omega t + a) - \frac{\omega_\alpha}{\omega}\right] \tag{11a}$$

and

$$J(t) = J_0 e^{-\omega_\alpha f(t)}, \tag{11b}$$

where a and J_o are constants determinate by the initial conditions, and $f(t)$ is defined as

$$f(t) = \frac{4(\tan \omega t - \omega_\alpha/\omega)^2 t}{1 + (\tan \omega t - \omega_\alpha/\omega)^2}. \tag{11c}$$

To know the trajectory in the new phase space (ϕ, J), one can express J as a function of ϕ through the integration of $\dot{J} = (dJ/d\phi)\dot{\phi}$. This brings about the following expression

$$J(\phi) = \frac{\tilde{J}_0}{\omega + \omega_\alpha \sin \phi} e^{\frac{2\omega_\alpha}{\omega} \arctan(\tan \phi + \omega_\alpha/\omega)}, \tag{12}$$

where $\tilde{J}_0$ is another constant. Figure 2 shows trajectories in this phase space for $\omega_\alpha = 0$ and for $\omega_\alpha \neq 0$, where the expected jumps at $\pi/2$ and $3\pi/2$ are clearly seen at the scale shown.

To finish the classical analysis, let us write the constant of motion, the Lagrangian and the generalized linear momentum for the weak dissipative case in terms of the variables ϕ and J. These are given by

$$K = \omega J + \omega_\alpha J \left[\sin(2\phi) - 2\phi\right], \tag{13a}$$

$$L = -\omega J \cos(2\phi) + 2\omega_\alpha J \left[\cos(2\phi) - 4 \sin(2\phi) \ln(\cos \phi)\right] \tag{13b}$$

and

$$p = \sqrt{2m\omega J} \sin \phi + \omega_\alpha \sqrt{\frac{2m J}{\omega}} \left[\cos \phi \left(1 - 2 \ln(\cos \phi)\right) - 2\phi \sin \phi\right] \tag{13c}$$

Once again, one see the impossibility to get the Hamiltonian $H(\phi, J)$ due to complexity of the expression (13c).

3. QUANTIZATION OF THE CONSTANT OF MOTION

Due to the impossibility of getting the Hamiltonian explicitly for the autonomous dynamical system (2), one may propose to extend the Shrödinger quantization to a dissipative system through the quantization of the constant of motion
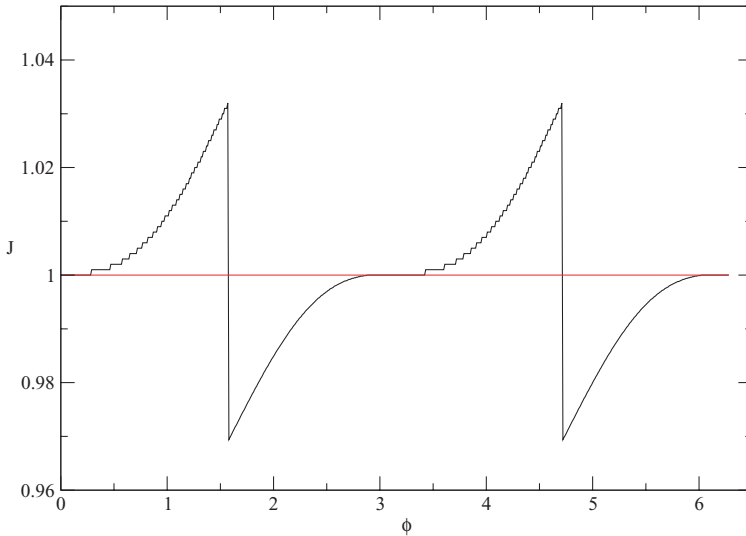
Fig. 2. Trajectory on the phase space (ϕ, J) as determinate by (12). The straight horizontal line corresponds to $\omega_\alpha = 0$, solid line corresponds to $\omega_\alpha = 0.001~\text{sec}^{-1}$, $\omega = 1~\text{sec}^{-1}$ and $\tilde{J}_0 = 1$ Joules-sec.

associated to it. This can be made by associating an Hermitian operator to the velocity as

$$\widehat{v} = -\frac{i\hbar}{m}\frac{\partial}{\partial x}. \tag{14}$$

In this way, if $\widehat{K}$ is the Hermitian operator associated to the constant of motion (which must have units of energy), the associated Shrödinger equation of the classical autonomous system would be

$$i\hbar\frac{\partial \Psi}{\partial t} = \widehat{K}(x, \widehat{v})\Psi, \tag{15}$$

where $\Psi = \Psi(x, t)$. Of course, the whole quantum mechanics structure is exactly the same but with the velocity operator instead of the linear momentum operator as the main operator of the quantum system. Since (15) represents an stationary problem, one just has to solve the eigenvalue problem

$$\widehat{K}(x, \widehat{v})\phi = E\phi, \tag{16}$$

where $\phi = \phi(x)$ and one has used $\Psi(x, t) = \phi(x)\exp(-iEt/\hbar)$ in (15). If $\widehat{K}$ can be written as $\widehat{K} = \widehat{K}_0 + \widehat{K}_I$, where the solution of the problem $\widehat{K}_0|n\rangle = E_n^{(0)}|n\rangle$ is known, the eigenvalues of (16) are given at first order in perturbation theory

(Messiah, 1958) by

$$E_n = E_n^{(0)} + \langle n \widehat{K}_I | n \rangle, \tag{17}$$

where one has used Dirac notation (Dirac, 1992). Our constant of motion (5) or (13a) can be expressed of the form $K = K_0 + K_I$, where $K_0 = \omega J$ and $K_I = \omega_\alpha J (\sin(2\phi) - 2\phi)$. It is well known that the eigenvalues of the harmonic oscillator without dissipation are given by $E_n^{(0)} = \hbar\omega(n + 1/2)$. Thus, $\widehat{K}_0$ is diagonal in the basis $\{|n\rangle\}$ and can be expressed in terms of ascent, a^+, and descent, a, operators as

$$\widehat{K}_0 = \hbar\omega(a^+ a + 1/2), \tag{18}$$

with a and a^+ defined in terms of $\widehat{x}$ and $\widehat{v}$ as

$$a = \sqrt{\frac{m}{2\hbar\omega}}(\omega\widehat{x} + i\widehat{v}), \quad \text{and} \quad a^+ = \sqrt{\frac{m}{2\hbar\omega}}(\omega\widehat{x} - i\widehat{v}). \tag{19}$$

These operator have the following commutation relations

$$[a, a^+] = 1, [a, a] = [a^+, a^+] = 0, [a, a^+ a] = a, [a^+, a^+ a] = -a^+.$$

Additionally, $\widehat{N} = a^+ a$ is called the number operator and is diagonal in the basis $\{|n\rangle\}$, $\widehat{N}|n\rangle = n|n\rangle$, This, in turns, implies that the operator associated to the variable J is given by

$$\widehat{J} = \hbar(\widehat{N} + 1/2). \tag{20}$$

Thus, our main problem is to assign a Hermitian operator to the function $K_1 = 2\omega_\alpha J (\cos\phi \sin\phi - \phi)$. According to refence (Leach, 1983), one has the following assignments

$$\sin\phi \longrightarrow \widehat{S}, \cos\phi \longrightarrow \widehat{C}, \phi \longrightarrow \widehat{\phi}, \tag{21}$$

where $\widehat{S}$ and $\widehat{C}$ are Hermitian operators such that

$$\ddot{\widehat{S}} + \omega^2 \widehat{S} = \ddot{\widehat{C}} + \omega^2 \widehat{C} = 0, \quad [\widehat{C}, \widehat{N}] = i\widehat{S}, \quad [\widehat{S}, \widehat{N}] = -i\widehat{C} \tag{22a}$$

and

$$\widehat{S}|n\rangle = \frac{i}{2}\Big(|n+1\rangle - |n-1\rangle\Big), \widehat{C}|n\rangle = \frac{1}{2}\Big(|n+1\rangle + |n-1\rangle\Big), [\widehat{C}, \widehat{S}] = \frac{\pi_0}{2i}, \tag{22b}$$

with π_0 is the projector on the ground state of the quantum harmonic oscillator, $\pi_0 = |0\rangle\langle 0|$. Therefore, $\widehat{C}$ and $\widehat{S}$ commute for any exited state. The operators $\widehat{\phi}$ is defined as

$$\widehat{\phi} = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \binom{-1/2}{k} \widehat{C}^{2k+1} \tag{23}$$

where $\binom{a}{b}$ si the combinatorial factor. In this way, the operator associated to the function K_I can be written as

$$\widehat{K}_I = \frac{\omega_\alpha}{3}\left[\widehat{J}\widehat{C}\widehat{S} + \widehat{C}\widehat{J}\widehat{S} + \widehat{C}\widehat{S}\widehat{J} + \widehat{J}\widehat{S}\widehat{C} + \widehat{S}\widehat{J}\widehat{C} + \widehat{S}\widehat{C}\widehat{J}\right] - \omega_\alpha\left[\widehat{J}\widehat{\phi} + \widehat{\phi}\widehat{J}\right] . \tag{24}$$

The first term has not contribution at first order in perturbation theory because of (22b), the action of the operators $\widehat{C}$ or $\widehat{S}$ increases or decreases the state number such that the expected value $\langle n|..|n\rangle$ is always zero. Due to the same reason, the contribution of the second term of Eq. (24) will come only from the term $\pi/2$ of (23). Thus, at first order in perturbation theory, one has the following correction of the eigenvalues

$$\delta E_n^{(1)} = -\hbar\omega_\alpha\pi(n + 1/2) . \tag{25a}$$

Thus, one gets

$$E_n \approx E_n^{(0)} + \delta E_n^{(1)} = \hbar\omega(n + 1/2)\left[1 - \frac{\omega_\alpha\pi}{\omega}\right] . \tag{26a}$$

That is, there is a small shift on the frequency of oscillation given by

$$\omega' = \omega - \pi\omega_\alpha . \tag{26b}$$

At second order in perturbation theory, $E_n = E_n^{(0)} + \delta E_n^{(1)} + \delta E_n^{(2)}$, the correction on the energy will come from the expression

$$\delta E_n^{(2)} = \sum_{k \neq n} \frac{|\langle k|\widehat{K}_I|n\rangle|^2}{E_n^{(0)} - E_k^{(0)}}. \tag{27}$$

Using (22b), (23) and (24), one has

$$\langle k|\widehat{K}_I|n\rangle = \frac{\hbar\omega_\alpha}{12}[(2k + 4n + 5)\delta_{k,n+2} - (2k + 4n + 1)\delta_{k,n-2}] + \hbar\omega_\alpha(n + k + 1)$$

$$\times \sum_{l=0}^{\infty}\sum_{s=0}^{2l+1} \frac{(-1)^l}{2l+1}\binom{-1/2}{l}\binom{2l+1}{s}\delta_{k,n-2l-1-s} \tag{28}$$

Therefore, the correction at second order can be written as

$$\delta E_n^{(2)} = -\frac{\hbar\omega_\alpha^2}{\omega}\left[\left(\frac{2}{3}n + \frac{1}{4}\right) - \sum_{l=0}^{\infty}\sum_{s=0}^{2l+1}\binom{-1/2}{l}^2\binom{2l+1}{s}^2\right.$$

$$\left.\times \frac{(2n - 2l - s)^2}{(2l+1)^2(2l+1+s)}\right] \tag{29}$$

4. CONCLUSIONS

We have used the constant of motion for the one-dimensional dissipative harmonic oscillator to study the classical trajectories in the phase space and to point out the difficulty to get its Hamiltonian explicitly. Due to this problem, we have proposed the quantization of the constant of motion directly, via the association of the velocity operator. Then, the eigenvalues were calculated up to second order within perturbation theory to see the first effect of the dissipation on them. This effect at first order corresponds to have a shift by the quantity $\pi\omega_\alpha$ on the frequency of oscillation of the nondissipative harmonic oscillator.

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